

6.8 Spherical Harmonics

Spherical harmonics occur in a large variety of physical problems, for example, whenever a wave equation, or Laplace's equation, is solved by separation of variables in spherical coordinates. The spherical harmonic $Y_{lm}(\theta, \phi)$, $-l \leq m \leq l$, is a function of the two coordinates θ, ϕ on the surface of a sphere.

The spherical harmonics are orthogonal for different l and m , and they are normalized so that their integrated square over the sphere is unity:

$$\int_0^{2\pi} d\phi \int_{-1}^1 d(\cos \theta) Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{l'l} \delta_{m'm} \quad (6.8.1)$$

Here asterisk denotes complex conjugation.

Mathematically, the spherical harmonics are related to *associated Legendre polynomials* by the equation

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi} \quad (6.8.2)$$

By using the relation

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi) \quad (6.8.3)$$

we can always relate a spherical harmonic to an associated Legendre polynomial with $m \geq 0$. With $x \equiv \cos \theta$, these are defined in terms of the ordinary Legendre polynomials (cf. §4.5 and §5.5) by

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \quad (6.8.4)$$

The first few associated Legendre polynomials, and their corresponding normalized spherical harmonics, are

$P_0^0(x) = 1$	$Y_{00} = \sqrt{\frac{1}{4\pi}}$
$P_1^1(x) = -(1-x^2)^{1/2}$	$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$
$P_1^0(x) = x$	$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta$
$P_2^2(x) = 3(1-x^2)$	$Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$
$P_2^1(x) = -3(1-x^2)^{1/2}x$	$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$
$P_2^0(x) = \frac{1}{2}(3x^2-1)$	$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2}\right)$

(6.8.5)

There are many bad ways to evaluate associated Legendre polynomials numerically. For example, there are explicit expressions, such as

$$P_l^m(x) = \frac{(-1)^m (l+m)!}{2^m m! (l-m)!} (1-x^2)^{m/2} \left[1 - \frac{(l-m)(m+l+1)}{1!(m+1)} \left(\frac{1-x}{2}\right) + \frac{(l-m)(l-m-1)(m+l+1)(m+l+2)}{2!(m+1)(m+2)} \left(\frac{1-x}{2}\right)^2 - \dots \right] \quad (6.8.6)$$

where the polynomial continues up through the term in $(1-x)^{l-m}$. (See [1] for this and related formulas.) This is not a satisfactory method because evaluation of the polynomial involves delicate cancellations between successive terms, which alternate in sign. For large l , the individual terms in the polynomial become very much larger than their sum, and all accuracy is lost.

In practice, (6.8.6) can be used only in single precision (32-bit) for l up to 6 or 8, and in double precision (64-bit) for l up to 15 or 18, depending on the precision required for the answer. A more robust computational procedure is therefore desirable, as follows:

The associated Legendre functions satisfy numerous recurrence relations, tabulated in [1-2]. These are recurrences on l alone, on m alone, and on both l and m simultaneously. Most of the recurrences involving m are unstable, and so dangerous for numerical work. The following recurrence on l is, however, stable (compare 5.5.1):

$$(l-m)P_l^m = x(2l-1)P_{l-1}^m - (l+m-1)P_{l-2}^m \quad (6.8.7)$$

It is useful because there is a closed-form expression for the starting value,

$$P_m^m = (-1)^m (2m-1)!! (1-x^2)^{m/2} \quad (6.8.8)$$

(The notation $n!!$ denotes the product of all *odd* integers less than or equal to n .)

Using (6.8.7) with $l = m+1$, and setting $P_{m-1}^m = 0$, we find

$$P_{m+1}^m = x(2m+1)P_m^m \quad (6.8.9)$$

Equations (6.8.8) and (6.8.9) provide the two starting values required for (6.8.7) for general l .

The function that implements this is

```

FUNCTION plgndr(l,m,x)
  INTEGER l,m
  REAL plgndr,x
  Computes the associated Legendre polynomial  $P_l^m(x)$ . Here  $m$  and  $l$  are integers satisfying
     $0 \leq m \leq l$ , while  $x$  lies in the range  $-1 \leq x \leq 1$ .
  INTEGER i,ll
  REAL fact,p1l,pmm,pmmp1,somx2
  if(m.lt.0.or.m.gt.l.or.abs(x).gt.1.)pause 'bad arguments in plgndr'
  pmm=1.                                Compute  $P_m^m$ .
  if(m.gt.0) then
    somx2=sqrt((1.-x)*(1.+x))
    fact=1.
    do 11 i=1,m
      pmm=-pmm*fact*somx2
      fact=fact+2.
    enddo 11
  endif
  if(l.eq.m) then
    plgndr=pmm
  else
    pmmp1=x*(2*m+1)*pmm                Compute  $P_{m+1}^m$ .
    if(l.eq.m+1) then
      plgndr=pmmp1
    else
      do 12 ll=m+2,l                    Compute  $P_l^m$ ,  $l > m+1$ .

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      pll=(x*(2*ll-1)*pmmp1-(ll+m-1)*pmm)/(ll-m)
      pmm=pmmp1
      pmmp1=pll
    enddo 12
    plgndr=pll
  endif
endif
return
END

```

CITED REFERENCES AND FURTHER READING:

- Magnus, W., and Oberhettinger, F. 1949, *Formulas and Theorems for the Functions of Mathematical Physics* (New York: Chelsea), pp. 54ff. [1]
- Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapter 8. [2]

6.9 Fresnel Integrals, Cosine and Sine Integrals

Fresnel Integrals

The two Fresnel integrals are defined by

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt, \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt \quad (6.9.1)$$

The most convenient way of evaluating these functions to arbitrary precision is to use power series for small x and a continued fraction for large x . The series are

$$\begin{aligned}
 C(x) &= x - \left(\frac{\pi}{2}\right)^2 \frac{x^5}{5 \cdot 2!} + \left(\frac{\pi}{2}\right)^4 \frac{x^9}{9 \cdot 4!} - \cdots \\
 S(x) &= \left(\frac{\pi}{2}\right) \frac{x^3}{3 \cdot 1!} - \left(\frac{\pi}{2}\right)^3 \frac{x^7}{7 \cdot 3!} + \left(\frac{\pi}{2}\right)^5 \frac{x^{11}}{11 \cdot 5!} - \cdots
 \end{aligned} \quad (6.9.2)$$

There is a complex continued fraction that yields both $S(x)$ and $C(x)$ simultaneously:

$$C(x) + iS(x) = \frac{1+i}{2} \operatorname{erf} z, \quad z = \frac{\sqrt{\pi}}{2}(1-i)x \quad (6.9.3)$$

where

$$\begin{aligned}
 e^{z^2} \operatorname{erfc} z &= \frac{1}{\sqrt{\pi}} \left(\frac{1}{z+} \frac{1/2}{z+} \frac{1}{z+} \frac{3/2}{z+} \frac{2}{z+} \cdots \right) \\
 &= \frac{2z}{\sqrt{\pi}} \left(\frac{1}{2z^2+1-} \frac{1 \cdot 2}{2z^2+5-} \frac{3 \cdot 4}{2z^2+9-} \cdots \right)
 \end{aligned} \quad (6.9.4)$$

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